Relativity in Velocity Space – Summary

1.) Part V of series of papers initiated at Chelsea by Donald Gillies – Parts I – III deal with early period in History & Philosophy of Non-Euclidean Geometry up to 1870's. Part IV is 'Last Stand of the Kantians' – deals with ~~Kant~~ Frege's reactionary Kantian views on Geometry. We now consider the Return of the Non-Euclideans in the context of S.R. (the original non-Euclidean theory Gauss, Riemann, Clifford etc)

2.) If by N-E Geometry we mean Geometry which is not (in some respect) Euclidean then first one in S.R. is Minkowski pseudo-Euclidean Geometry of space-time (1908). But this does not effect a transition from Euclidean to N-E Geometry since the Geometry of Newtonian spacetime (under the Galilean group) is certainly not Euclidean. (4-dimensional) – involves a shearing motion rather than a rotation under Galilean boosts.

3.) S.R. the spatial Geometry at a given time is Euclidean – it is only with G.R. that we come up against N-E Geometry of 3-dimensional space – first hint was implicit in Ehrenfest's 1909 paper on ~~showing~~ measurements on a rigid rotating disc – developed by Einstein in period 1913-1916 ~~(Taken up Ehrenfest's point in 1911)~~

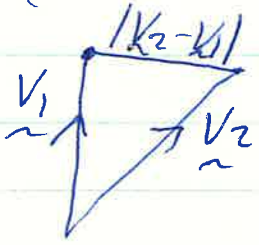
NB By patching together position space
, velocity space we get extended
phase space

4.) In Newtonian physics Euclidean geometry enters twice over
a) It is the geometry of physical space
b) It is the geometry of velocity space.

b) follows since the homogeneous Galilean group includes rotations and displacements in velocity space.

Note that thus any two velocities are connected by a displacement just as any two positions are (for a single particle)

the metric in velocity space is just the magnitude of the relative velocity



cf $\underset{\sim}{x}' = \underset{\sim}{x} + \underset{\sim}{v}t$
$t' = t$
$\Longrightarrow \underset{\sim}{u}' = \frac{d\underset{\sim}{x}'}{dt'} = \underset{\sim}{u} + \underset{\sim}{v}$

5) The transition to S.R. lies in the fact that the geometry of velocity space is non-Euclidean - indeed it is Lobachevskian with Gaussian curvature $-1/c^2$.

We shall now explain this and then give the history.

Consider two points on x-axis in vel. space with velocities u_1, u_2
Under Lorentz boosts v along the x-axis let new velocities be w_1 and w_2.

then Einstein's velocity addition result yields

$$w_1 = \frac{u_1 + v}{1 + u_1 v}$$

$$w_2 = \frac{u_2 + v}{1 + u_2 v}$$

Write $\rho_v = th^{-1} v = \frac{1}{2} \ln \left(\frac{1+v}{1-v}\right)$ (ρ_v is rapidity variable associated with speed v)

then we have $\rho_{w_1} = \rho_{u_1} + \rho_v$

$$\underline{\rho_{w_2} = \rho_{u_2} + \rho_v}$$

∴ by subtraction

$$\rho_{w_1} - \rho_{w_2} = \rho_{u_1} - \rho_{u_2}$$

and we can accordingly define an invariant metric

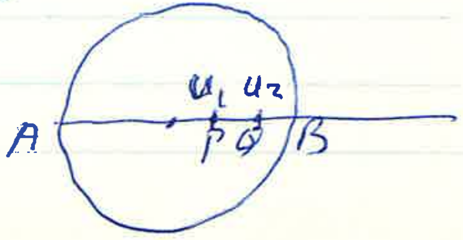
$$d(u_1, u_2) = |\rho_{u_1} - \rho_{u_2}|$$

$$= \frac{1}{2} \left| \ln \left(\frac{1+u_1}{1-u_1}\right)\left(\frac{1-u_2}{1+u_2}\right) \right|$$

$$= \frac{1}{2} \left| \ln . \{-1, +1 ; u_1, u_2\} \right|$$

where we define cross-ratio of four parameters $a\, b\, c\, d$ by $\{a, b ; c, d\} = \frac{(a-c)(b-d)}{(a-d)(b-c)}$

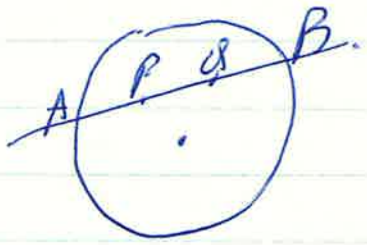
So we can give a geometrical picture for this metric



$d(u_1, u_2) = \frac{1}{2} |\ln \{AB, PQ\}|$

This suggests for points set on x-axis we define

$$d(u_1, u_2) = \tfrac{1}{2}\left|\ln\{AB, PQ\}\right|$$

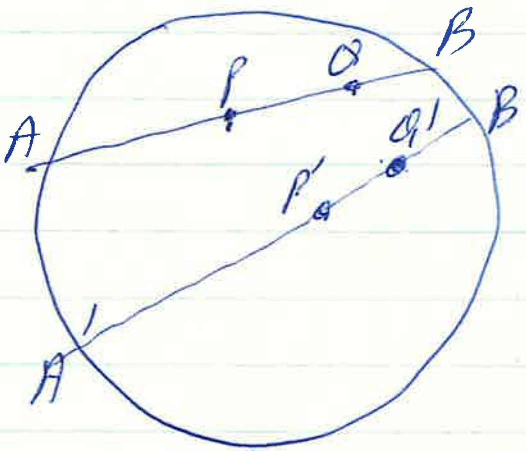


where $\{AB;PQ\} = \dfrac{AP}{AQ} \cdot \dfrac{QB}{PB}$

$$= \frac{AP}{PB} \Big/ \frac{AQ}{QB}$$

Now the transformations induced in velocity space by the homogeneous Lorentz group are projective transformations (ratios of linear transformations) of velocity components which leave the light sphere invariant.

But cross-ratio is invariant under projective transformations so $d(P,Q) = d(P',Q')$ thus



A must project to A' and B to B' since eg. A', B' must lie on P'Q' produced and also on the invariant light circle.

The geometry induced on points interior to the light sphere is well known to be a model for Lobachevsky's geometry (Klein 1871) note how the 'distance' to the light sphere becomes infinite in this geometry.

6) We can also ask what geometry is induced on the light-sphere. The answer is 2-dimensional conformal geometry — this leads rather directly to the representation of the Lorentz gp transformations in terms of a) quaternions and b) spinors and hence indirectly to the Dirac eq. (1928)

7.) The history is briefly as follows:

Sommerfeld 1909. 'On the Composition of velocities in Relativity'
interprets velocity addition as spherical trigonometry in terms of imaginary angles.

$\theta = i\,\Omega\,v$

leads to Varičak 1910. 'Application of Lobatschefskian geometry in Relativity'
— does not deal with points in vel. space etc — only velocity addition in terms of hyperbolic △s.

Independent treatment including geometry of light sphere by

Herglotz 1910. "On those bodies which may be called rigid from the standpoint of the relativity principle"

The relation with projective geometry is used here

The link with projective geometry is widely identified by Pauli in his 1921 Encyclopedia article — no acknowledgement of the Herglotz paper.

The work of Pauli & Herglotz 'rediscovered' by J. Le Roux (1922) 'Special Relativity and the Geometry of Wave Systems'.

The Cayley nature was also 'rediscovered' independently by

Malvaux (1952) "Researches on an invariance law for the composition of velocities'.

Finally Fock in his book 'The Theory of Space Time & Gravitation (1955) developed the idea of an invariant relative velocity (essentially $|V_1 - V_2| = th(\Lambda v_1 - \Lambda v_2)$) and expresses the Lobachevsky geometry of velocity space in terms of these relations.

(cf also Abelé & Malvaux. — Vitesse et Univers Relativiste 1954 — no reference to earlier work)

See also Liebscher (1978) "Derivation of Einstein's Velocity Addition Theorem through use of the invariant double ratio"

N.B. rotators are incompatible w. S.R.
since they are not 4-vectors.

(VI) An appropriate sub-group of the multiplicative group
in of quaternion algebra

8) Conclusions

A velocity space is a natural framework wherein to ~~[illegible]~~ interpret the notions of Einsteinian relativity — it leads effectively to establishing the isomorphism of the homogeneous Lorentz group with other groups such as

(i) the motions in hyperbolic geometry
(ii) The conformal transformations of a 2-dimensional sphere

(iii) The bilinear transformations of a complex plane

(iv) the projective transformations of 1-dimensional complex projective space

(v) the unimodular linear transformations of a complex 2-dimensional vector space. (homomorphism with Lorentz group.)

Herein lie all the deep insights into the Lorentz group.

B Brouwer 1909 inaugural lecture hailed the discovery of the applicability of hyperbolic geometry as challenging Kant's ~~a priori~~ synthetic a priori account of the propositions of Euclidean geometry.

But does this itself really challenge Kant? ∴ since props of geometry are synthetic the denial of Eucl. geometry is not logically inconsistent for Kant while their a priori character is in respect of their applicability to the forms of (spatial) intuition 'but this refers to physical position space, not presumably to ~~[illegible]~~ an 'artificial' velocity space.

[illegible] quote

"Here {in vel. space} the transition from a 3-dimensional ~~geometry~~ Euclidean geometry to a Non-Euclidean Geometry has actually taken place and therefore it is no longer possible to maintain that the other original Euclidean Geometry" of experience is a priori in physics.

ambiguity – does B
mean other branches
of physics.

So do we have to wait for N-E geometry of physical space as in G-R to challenge Kant's claims?